

Consider the linear system

$$Ax = b$$

When does it have solutions? If it has solutions, is it unique?

Two special cases:

1. **When A has full column rank**; that is A is a tall matrix and it is an overdetermined system
 - a. **If b is not in the range space of A, then there does not exist a solution.** This is obvious according to $Ax = b$.
 - b. **If b is in the range space of A, then there exists a unique solution,**

$$x = (A^T A)^{-1} A^T b$$

PS: $Ax = A(A^T A)^{-1} A^T b$ is the orthogonal projection of b onto the range space of A. In fact, no matter if b is in the range space of A, we can always have this value, but we should clearly know when it is a solution to the linear system

2. **When A has full row rank** – That is A is a wide matrix and it is an underdetermined system
 - a. **The system $Ax = b$ always has solutions and the solution is not unique if A is not square.** A solution is

$$x = A^T (AA^T)^{-1} b$$
 It is obvious that it satisfies the linear system. Since A is wide, its null space is nontrivial and the general expression of the solution is $x = A^T (AA^T)^{-1} b + y$ where y is an arbitrary vector in the null space of A
3. When A is square and nonsingular, it is the most special and simplest case.

A general and difficult problem is what if A does not have full row or full column rank. What will the solution be? To solve this problem, we use SVD

Let $A = U\Sigma V^T$ be the SVD of A. then

$$Ax = b \Leftrightarrow U\Sigma V^T x = b \Leftrightarrow \Sigma \bar{x} = \bar{b}$$

1. If $\Sigma \rightarrow \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$; that is **A does not has full column or full row rank**. This is the most general case

$$Ax = b \Leftrightarrow \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_0 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_0 \end{bmatrix} \Leftrightarrow \Sigma \bar{x}_1 = \bar{b}_1 \text{ and } \bar{b}_0 = 0$$

If $\bar{b}_0 \neq 0$, then it has not solutions. Otherwise, when $\bar{b}_0 = 0$, the system has solutions and the solution is not unique: $\bar{x}_1 = \Sigma^{-1} \bar{b}_1$ and \bar{x}_0 can be arbitrary. **In summary, if b is in the range space of A, then it has solutions and the solution is not unique.**

A conclusion is that for the system $Ax = b$, it always has a solution for all b if and only if A has full row rank. Otherwise, the system may have solutions for some b, and no solutions for the other b. Furthermore, if A is square, then the solution is unique; if A is wide, then the solution is not unique.