

Consider the discrete time dynamical system

$$x_{k+1} = Ax_k + Bu_k$$

Definition of controllability: The system is defined as controllable if there exists a control sequence u_0, u_1, \dots, u_{k-1} that can steer the state from any x_0 to any x_k .

Condition of controllability: The matrix $[B, AB, A^2B, \dots, A^{k-1}B]$ has full row rank.

Proof:

$$\begin{aligned} x_1 &= Ax_0 + Bu_0 \\ x_2 &= Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1 \\ x_3 &= Ax_2 + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2 \\ &\dots \\ x_k &= A^kx_0 + A^{k-1}Bu_0 + A^{k-2}Bu_1 + \dots + ABu_{k-1} + Bu_k \end{aligned}$$

Therefore,

$$\underbrace{x_k - A^kx_0}_b = \underbrace{[A^{k-1}B \quad A^{k-2}B \quad \dots \quad AB \quad B]}_E \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \dots \\ u_{k-1} \\ u_k \end{bmatrix}}_u$$

Which can be written as

$$b = Eu$$

According to the definition of controllability, there should exist u for arbitrary b (i.e., arbitrary x_k and x_0). Then, by the properties of the linear system, we know E must have full row rank. QED

Based on the linear system, we can understand all kinds of questions regarding the controllability.

- Question:** In some literature, the controllability is defined as from $x_0 = 0$ to arbitrary x_k . Is it the same definition as from any x_0 to any x_k ?

Answer: Yes. The key problem is when $b = x_k - A^kx_0$ is arbitrary. If $x_0 = 0$ and the system can reach arbitrary x_k , then for arbitrary x_0 , the system can also reach arbitrary x_k . The converse is also true.
- Question:** Does controllability mean the control is unique?

Answer: In general, no. From the linear system, we know if E is a square matrix, then the control is unique. Otherwise, if E is wide, then the control is not unique.
- In some literature, the definitions of controllability are different. But we can understand them by analyzing the linear system

any $x_0 \rightarrow 0$: a state x_0 is controllable if it can be controlled to the origin. A linear system is (completely) controllable if all states can be controlled to origin. (the system is controllable if E has full row rank. But even if E does not have full row rank, it can also be completely controllable because $A^kx_0 \in \text{Range}(A)$ for all x_0 can be in the range space of E even if E)

$0 \rightarrow$ any x_k : A state x_k is reachable if the state can be reached from the origin. A linear system is (completely) reachable if all the states can be reached from the origin. (the system is reachable if and only if E has full row rank)

any $x_0 \rightarrow$ any x_k : The definition of controllability given in Fabio paper and T. Kailath's book actual is $x_0 \rightarrow x_k$ (i.e., from any initial condition to any final state). This definition is the same as $0 \rightarrow x_k$ (the system is controllable if and only if E has full row rank)