We often encounter rotating one vector. What if we rotate a reference frame to coincide with another reference frame? What will we get?

The 2D rotation matrix which obeys right-hand rule is

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

The 3D rotation matrices that obey right-hand rule are

$$\mathrm{R_x} = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{bmatrix}, R_y = \begin{bmatrix}\cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta\end{bmatrix}, R_z = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}$$

**Claim: Rotate frame 1 to coincide with frame 2, we can get a rotation matrix R. Then this R is the rotation form 2 to 1.**

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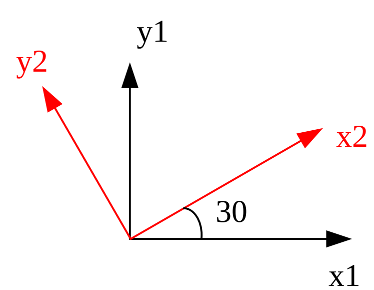
**Example A:**




The standard 2D rotation matrix is

$$R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

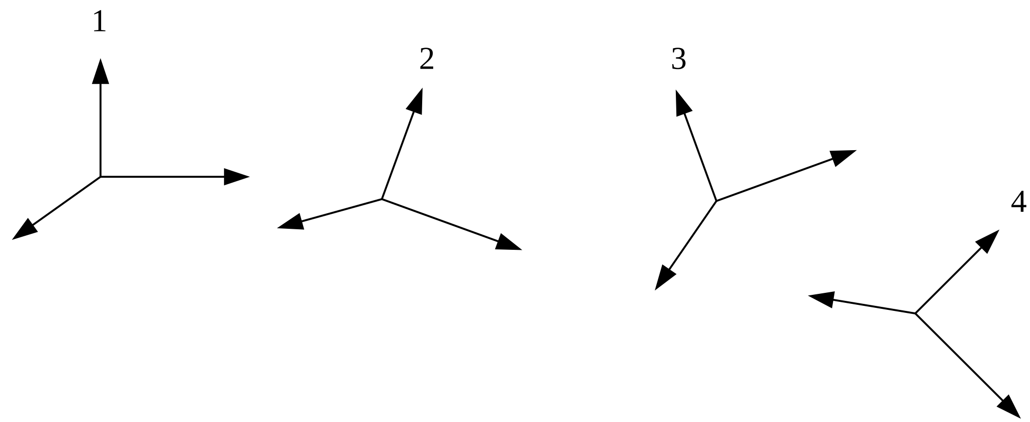
Obviously, rotate frame 1 30 degree anti-clockwise, we can get frame 2. Rotating 30 degree anti-clockwise gives $R(\theta = 30)$.

By observation,

$$p_1 = R(\theta = 30)\mathrm{p}_2$$

So $R(\theta = 30)$ is the rotation from frame 2 to frame 1.

**Example B:**




Here we only consider rotations.

i) Rotate frame 1 to coincide with frame 2, we actually get $\mathrm{R}_{1/2}$.
ii) Rotate frame 2 to coincide with frame 3, we actually get $\mathrm{R}_{2/3}$.
iii) Rotate frame 2 to coincide with frame 3, we actually get $\mathrm{R}_{3/4}$.

So rotate frame 1 to coincide with frame 2, then frame 3, then frame 4, we get

$$\mathrm{R}_{1/2}\mathrm{R}_{2/3}\mathrm{R}_{3/4}$$

which equals $\mathrm{R}_{1/4}$. Note the sequence.

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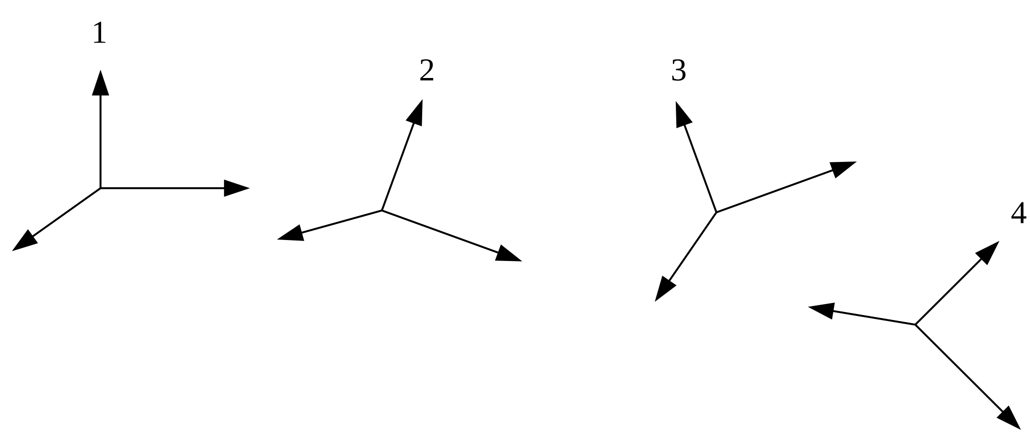
**Example 3: Euler angles**



The key to understand Euler angles is the intermediate frames. We have frame 1 to frame 4. If we rotate frame 1 to coincide with frame 4, then we get a rotation matrix. That matrix is the rotation from frame 4 to frame 1. What will that matrix be? Using the example 2, very easy!

i) Rotate frame 1 to coincide with frame 2, we get a rotation matrix. Since the rotation about the z axis, so the rotation matrix is $R_z$. And this matrix actually is $R_{1/2}$.
ii) Rotate frame 2 to coincide with frame 3, we get a rotation matrix. Since the rotation about the y axis, so the rotation matrix is $R_y$. And this matrix actually is $R_{2/3}$.
iii) Rotate frame 3 to coincide with frame 4, we get a rotation matrix. Since the rotation about the x axis, so the rotation matrix is $R_x$. And this matrix actually is $R_{3/4}$.

Finally we have

$$R_{1/4} = R_{1/2}R_{2/3}R_{3/4} = R_z R_y R_x = R_z(\psi)R_y(\theta)R_x(\phi)$$

Then we have

$$R_{1/4} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

The fact is

$$R_{n/b} = R_\psi R_\theta R_\phi$$
$$= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

So frame 1 is frame n, frame 4 is frame b. The conclusion is

**Rotate the navigation frame to coincide with the body frame, the sequence is to rotate about z, y and x axis. The rotation angles are the Euler angles!!!**

**Keep the above in mind. These are the Euler angles we use in aerospace engineering.**

**If we rotate the body frame to coincide the navigation frame, what we get is $R_{b/n}$. It is not appropriate.**