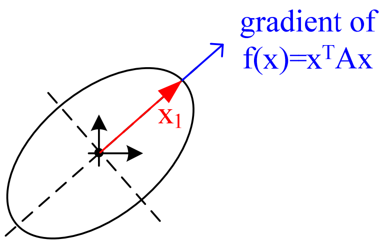
In R^n space, A is a positive definite matrix, then

$$x^T A x = k$$

stands for an ellipsoid.

Statement: The ellipsoid has n axes, then the n axes are respectively the eigenvector of A. Moreover, the length of the semi-axis is

$$\sqrt{\frac{k}{\lambda_i}}$$



Proof: The key of the proof is you must understand the definition of axis. As shown above, the axis x_1 must be collinear with the gradient of $f(x) = x^T A x$, which is

$$\nabla_x f(x) = 2Ax$$

Since $2Ax_1$ is collinear with x_1

$$2Ax_1 = \alpha x_1$$

i.e., x_1 is an eigenvector of A.

If $x_1^T A x_1 = k$, then

$$x_1^T A x_1 = \lambda_1 x_1^T x_1 = \lambda_1 \|x_1\|^2 = k$$

So we have

$$\|x_1\| = \sqrt{\frac{k}{\lambda_1}}$$

In addition:

1) Usually a ellipsoid is given as $x^T A x = 1$, then the length of the axis is $\sqrt{\frac{1}{\lambda_1}}$

2) The volume of a ellipsoid is $\frac{4}{3}\pi abc$, then the volume is

$$V = \frac{4}{3}\pi \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}}$$

3) The most general ellipsoid is given by

$$(x - v)^T A (x - v) = k$$